

## Chiral Symmetry breaking in Bosonic Partition Functions

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The behavior of quenched Dirac spectra of two-dimensional lattice QCD is consistent with spontaneous chiral symmetry breaking which is forbidden according to the Coleman-Mermin-Wagner theorem. One possible resolution of this paradox is that, because of the bosonic determinant in the partially quenched partition function, the conditions of this theorem are violated allowing for spontaneous symmetry breaking in two dimensions or less. This goes back to work by Niedermayer and Seiler on nonamenable symmetries of the hyperbolic spin chain and earlier work by two of the authors on bosonic partition functions at nonzero chemical potential. In this talk we discuss chiral symmetry breaking for the bosonic partition function of QCD at nonzero isospin chemical potential and a bosonic random matrix theory at imaginary chemical potential and compare the results with the fermionic counterpart. In both cases the chiral symmetry group of the bosonic partition function is noncompact.

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## 1. Introduction

According to the celebrated Coleman-Mermin-Wagner theorem, continuous symmetries cannot be broken spontaneously in two or less dimensions. In essence, the reason is that because of the fluctuations, the order parameter averages to zero. However this theorem does not apply to non-compact symmetries. This was pointed out by Niedermaier and Seiler [1], who argued that nonamenable symmetries are necessarily broken spontaneously in two dimensions or less. Nonamenable Lie groups are Lie groups for which no invariant mean exists such as for example noncompact semi-simple Lie groups – it cannot exist because of the divergent group volume. As emphasized in particular by Seiler [2] spontaneous symmetry breaking of noncompact symmetries is unavoidable in any dimensions of space-time.

Noncompact symmetries are an essential ingredient of the spectral analysis of disordered systems. The reason is that the resolvent is given by the derivative of a ratio of determinants, e.g. in QCD

$$G(z) = \text{Tr} \left\langle \frac{1}{D+z} \right\rangle = \frac{d}{dz'} \bigg|_{z'=z} \left\langle \frac{\det(D+z')}{\det(D+z)} \det^{N_f}(D+m) \right\rangle, \quad (1.1)$$

where  $D$  is the anti-Hermitian QCD Dirac operator and  $N_f$  is the number of flavors with quark mass  $m$ . Let us consider the simplest case, which is the quenched limit ( $N_f = 0$ ). Then the bosonic determinant can be represented as (for  $\text{Re}(z) > 0$ )

$$\frac{1}{\det(D+z)} = \int d\phi_1 d\phi_2 d\phi_1^* d\phi_2^* \exp \left[ - \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}^T \begin{pmatrix} z & id \\ id^\dagger & z \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right]. \quad (1.2)$$

The axial symmetry is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \rightarrow \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \quad (1.3)$$

with  $s$  real. Note that the  $U(1)$  symmetry of the action is not a symmetry of the partition function because such transformations will violate the complex conjugation property of the  $\phi_k$  which is required to have convergent integrals. Rather we observe that the axial symmetry group is non-compact and given by  $Gl(1)/U(1)$  [3, 4, 5]. This argument can be easily extended to include  $N_f$  flavors. In the case of the fermionic partition function, which can be represented as the average of a Grassmann integral, convergence is not an issue and the axial symmetry group can be chosen to be the compact  $U(1)$  group. The noncompact transformation is a symmetry group as well but would lead to divergent contributions if pushed forward onto the bosonized degrees of freedom (see [6]).

In [8] we analyzed Dirac spectra of the naive QCD Dirac operator in two dimensions, and found the same degree of agreement with random matrix predictions as was the case for QCD in four dimensions [8]. One possible explanation could be that states are localized but that the localization length is much larger than the size of the box. However, the volumes considered [8] were too small to confirm this possibility.

Alternatively, according to the arguments of Niedermaier and Seiler, the axial symmetry group, because of its noncompactness, is always broken spontaneously also in dimensions of two or less,

and it is possible to have extended quantum states. However, the conditions for this statement to be valid remain unclear. For example, chiral symmetry may be restored in the presence of an external field such as an (imaginary) chemical potential, and this is also expected to occur for a bosonic partition function. We also note that it is possible to have a nonzero density of states near zero without spontaneous symmetry breaking but rather with localized wave-functions [7]. What happens depends on the renormalization group flow but this is not within the scope of this work.

In this paper we address a simpler question: we explore the difference between fermionic and bosonic partition functions. We consider QCD at nonzero isospin chemical potential as well as a random matrix model at nonzero (imaginary) chemical potential [9, 10], and analyze the phase diagram of the corresponding fermionic and bosonic partition function.

## 2. Phase Quenched QCD

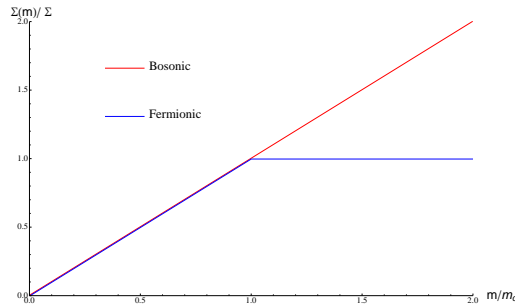
Some time ago we analyzed [11] the difference between the fermionic and bosonic partition function in the  $\varepsilon$ -domain of phase quenched QCD at nonzero chemical potential. The fermionic partition function given by

$$\langle \det(D + m + i\mu\gamma_0) \det(D + m - i\mu\gamma_0) \rangle. \quad (2.1)$$

This partition function is well understood [13]. It is QCD at nonzero isospin chemical potential with pion condensation for  $\mu > m_\pi/2$ . In the  $\varepsilon$ -domain of the QCD partition given by an integral over SU(2)

$$Z = \int_{U \in \text{SU}(2)} \exp\left[-\frac{1}{4}V\mu^2 F_\pi^2 \text{Tr}[U, \tau_3][U^{-1}, \tau_3] + \frac{1}{2}V\Sigma \text{Tr}M(U + U^{-1})\right]. \quad (2.2)$$

The SU(2) flavor symmetry of the fermionic partition function is broken to U(1) by the chemical potential, and the residual U(1) symmetry is broken spontaneously by the formation of a pion condensate. In the normal phase for  $\mu < m_\pi/2$ , the saddle point of the chiral Lagrangian is at  $U = 1$  while for  $\mu > m_\pi/2$  we are in a condensed phase with one exactly massless Goldstone boson [13]. The partition function is  $\mu$ -independent for  $\mu < m_\pi/2$  with a mass independent chiral condensate while it increases linearly in  $m$  for  $m_\pi/2 < \mu$  (see Fig. 1 for mean field results).



**Figure 1:** The mean field result for the mass dependence of the normalized chiral condensate for the phase quenched bosonic (red) and fermionic (blue) QCD partition function at nonzero chemical potential  $\mu$  versus the quark mass in units of the critical quark mass,  $m_c$ , for which a condensation transition occurs.

The corresponding bosonic partition function is given by

$$\left\langle \frac{1}{\det(D + m + i\mu\gamma_0) \det(D + m - i\mu\gamma_0)} \right\rangle. \quad (2.3)$$

When  $\lambda$  is an eigenvalue of  $D + i\mu\gamma_0$ , then  $\lambda^*$  is an eigenvalue of  $D - i\mu\gamma_0$ . Expressing the average as an integral over the spectral density, this results in a logarithmic divergence for  $\lambda \rightarrow m$

$$\int_{C_\varepsilon(m)} \frac{d\lambda d\lambda^*}{(\lambda - m)(\lambda^* - m)} \sim \log \varepsilon, \quad (2.4)$$

where  $C_\varepsilon(m)$  is the region between a unit circle centered at  $m$  and a circle of radius  $\varepsilon$  also centered at  $m$ . This divergence of the partition function can be regularized as

$$\left\langle \det^{-1} \begin{pmatrix} D + m + i\mu\gamma_0 & \varepsilon \\ \varepsilon & D + m - i\mu\gamma_0 \end{pmatrix} \right\rangle. \quad (2.5)$$

Using this regularization, the static part of the bosonic phase quenched partition function takes the form [14]

$$Z = \int_{Q \in \text{Gl}(2)/\text{U}(2)} \frac{dQ}{\det^2 Q} \theta(Q) \exp\left[-\frac{1}{4} V F_\pi^2 \mu^2 \text{Tr}[Q, \tau_3][Q^{-1}, \tau_3] + \frac{i}{2} V \Sigma \text{Tr} M(Q - I Q^{-1} I)\right], \quad (2.6)$$

with  $M = \varepsilon + m\tau_1$  and  $I = -i\tau_2$ . After the transformation  $Q = i\tilde{Q}\tau_1$  the partition function can be written as

$$Z = \int_{-i\tau_2 \tilde{Q} \in \text{Gl}(2)/\text{U}(2)} \frac{d\tilde{Q}}{\det^2 \tilde{Q}} \theta(i\tilde{Q}\tau_1) \exp\left[-\frac{1}{4} V F_\pi^2 \mu^2 \text{Tr}[\tilde{Q}, \tau_3][\tilde{Q}^{-1}, \tau_3] - \frac{1}{2} V \Sigma \text{Tr}(M_\varepsilon \tilde{Q} + M_{-\varepsilon} \tilde{Q}^{-1})\right]$$

with  $M_\varepsilon = M\tau_1$ . Apart from the measure, for  $\varepsilon \rightarrow 0$  the bosonic and fermionic chiral Lagrangian have the same functional dependence. At nonzero  $\mu$  the  $\text{U}(2)$  symmetry of the bosonic partition function is also broken to  $\text{U}(1)$ . This residual  $\text{U}(1)$  symmetry is broken spontaneously by the condensate resulting in a massless mode. For the bosonic theory this is the case for any value of  $\mu$  while for the fermionic theory the  $\text{U}(1)$  symmetry is restored for  $\mu < m_\pi/2$ .

The phase transition of the fermionic partition function occurs at the point where the saddle point hits the boundary of the manifold. In the bosonic case there is no boundary and the normal phase does not occur. Therefore, for any  $\mu > 0$  we have a massless charged boson with nonzero isospin charge that condenses. In Fig. 1 we show the mass dependence of the chiral condensate for both partition functions. In the fermionic case, there is a phase transition at  $m = m_c$  which is the quark mass for which  $\mu = m_\pi/2$ , while in the bosonic case there is no such transition.

In terms of  $\tilde{Q}$  the charged states reside in its off-diagonal matrix elements, which are the diagonal matrix elements of  $Q$ . Indeed they contain a massless mode parameterized by  $Q \rightarrow \exp[\tau_3 \alpha] Q \exp[\tau_3 \alpha]$  in the parameterization of [14].

What we conclude from this example is that spontaneous symmetry breaking can persist in the bosonic theory while it is restored in the fermionic theory for the same value of the parameters.

### 3. Bosonic versus Fermionic one-flavor Partition Functions

In this section we consider the one-flavor QCD partition function. Before analyzing the random matrix model at nonzero imaginary chemical potential, we first remind the reader of the  $\varepsilon$ -limit of the bosonic and fermionic partition functions of one-flavor QCD at  $\mu = 0$ .

### 3.1 One Flavor Partition Function at Zero Chemical Potential

At fixed topological charge  $\nu$ , the  $\varepsilon$ -limit of the QCD partition function has a residual  $U(1)$  covariance so that it is given by [15, 16]

$$Z_v^{N_f=1}(m) = \int_{U \in U(1)} dU \det^\nu U e^{\frac{1}{2} m V \Sigma \text{Tr}(U+U^{-1})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i\nu\theta} e^{mV\Sigma \cos \theta} = I_\nu(mV\Sigma). \quad (3.1)$$

The bosonic partition function is obtained by replacing the  $U(1)$  integral by a  $Gl(1)/U(1)$  integral and is thus given by

$$Z_v^{N_f=-1} = \int_{U \in Gl(1)/U(1)} dU \det^\nu U e^{-mV\Sigma \text{Tr}(U+U^{-1})} = \int_{-\infty}^{\infty} ds e^{\nu s} e^{-mV\Sigma \cosh s} = K_\nu(mV\Sigma). \quad (3.2)$$

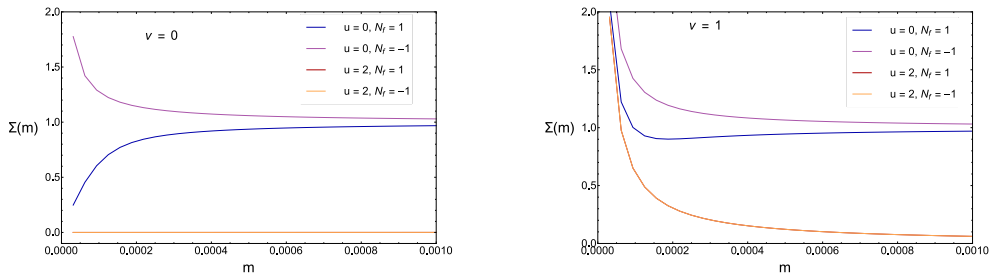
In both cases, for  $\nu \neq 0$  the chiral condensate diverges as  $|\nu|/m$  for  $m \rightarrow 0$ . For  $\nu = 0$ , the fermionic condensate vanishes in the chiral limit while the bosonic condensate diverges as  $1/m$ . At fixed  $\theta$  angle the condensates in each sector are weighted by  $Z_\nu/Z$  so that in the chiral limit only the contributions for  $\nu = \pm 1$  are nonvanishing in the fermionic case. In the bosonic case, the sum over  $\nu$  is divergent. Presently, it is not clear if the bosonic partition function can be defined at fixed  $\theta$ .

### 3.2 Chiral Random Matrix Theory at Imaginary Chemical Potential

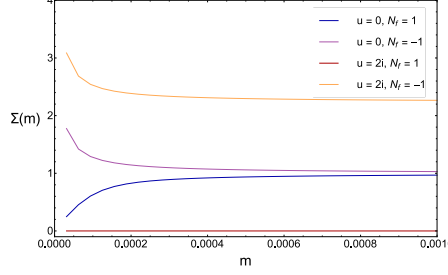
In this subsection we study a chiral random matrix model at imaginary chemical potential. The Dirac operator is given by [9, 10]

$$D = \begin{pmatrix} z & id + iu \\ id^\dagger + iu & z \end{pmatrix}. \quad (3.3)$$

The partition function is given by the expectation value of the determinant of  $D$  averaged over Gaussian distributed matrix elements  $d$ . In general,  $d$  is a  $n \times (n + \nu)$  matrix so that it has  $\nu$  zero modes. In the thermodynamic limit we keep  $N \equiv 2n + \nu$  fixed. This model was first studied in [9] where it was shown that it has a second order chiral phase transition with mean field critical indices. For imaginary  $u$ , or real chemical potential, the partition function corresponding to  $D$  is a model for QCD at nonzero baryon chemical potential with a phase transition to a phase with a nonzero baryon density [12]. If  $u$  is interpreted as the lowest Matsubara frequency, it is also a mean field model for the chiral phase transition as a function of temperature [9]. The normalization can



**Figure 2:** The mass dependence of the chiral condensate for the bosonic and fermionic partition function for  $\nu = 0$  (left) and  $\nu = 1$  (right). We show results for  $u = 0$  and  $u = 2$ .



**Figure 3:** The mass dependence of the chiral condensate of real and imaginary chemical potential (see legend). The chiral condensate of the bosonic partition function remains nonzero also at large values of the chemical potential (orange curve).

be chosen such that the chiral condensate is normalized to one for  $u = 0$  and becomes zero at  $u = 1$ . For imaginary  $u$ , using the same units, the phase transition is at  $u = 0.52$  though [12].

The fermionic partition function was worked out in [9, 17, 10]. For one-flavor the expression for finite  $n$  can be simplified to

$$Z_v^{N_f=1}(m, u) = \int_0^\infty ds s^{v+1} I_v(2nms\Sigma)(s^2 + u^2)^n e^{-n(s^2 + m^2)}.$$

For  $u < 1$ , the chiral condensate is given by  $\Sigma(u) = \sqrt{1 - u^2}$  while for  $u > 1$  the condensate vanishes in the chiral limit (both for  $v = 0$ ). For small nonzero  $m$  it is given by  $\Sigma(u) \sim mu/\sqrt{u^2 - 1}$ .

The bosonic partition function can be reduced to a one dimensional integral [18]

$$Z_v^{N_f=-1}(m, u) = m^{-v} \int_0^\infty \frac{ds}{s^{v+1}} e^{-snm^2/2} \frac{1}{(1/s + 1/\Sigma^2)^{n+v}} e^{-nu^2/(1/s + 1/\Sigma^2)}. \quad (3.4)$$

This partition function also shows a phase transition to a chirally restored phase at  $u = 1$ . For  $u < 1$  we find the same expression for the chiral condensate as in the fermionic case but for  $u > 1$  we obtain a slightly different result,  $\Sigma(u) \sim m/\sqrt{u^2 - 1}$  (also for  $v = 0$ ). The main difference between the fermionic and the bosonic partition function is that for  $v = 0$  the chiral condensate in the first case vanishes for  $m \rightarrow 0$  at fixed  $n$ , while in the case of the bosonic partition function it diverges as  $1/m$  for  $m \rightarrow 0$  at fixed  $n$ . In this sense the chiral symmetry of the bosonic partition function can also be broken in the chiral limit.

A second difference between the bosonic and fermionic partition function concerns its properties under analytical continuation. Replacing  $u \rightarrow i\mu$  in Eq. (3.4) reproduces the result for the random matrix model at nonzero real chemical potential [12, 17]. The bosonic partition function can also be continued naively to imaginary  $u$ . In Fig. 3 we show that in this case the chiral condensate remains finite for parameter values for which the chiral condensate of the fermionic partition function vanishes in the thermodynamic limit. Whether the analytical continuation in  $u$  is valid for the bosonic partition function as well remains to be determined [18].

## 4. Conclusions

We have compared spontaneous chiral symmetry breaking for bosonic and fermionic partition functions, and have studied phase quenched QCD and a one-flavor random matrix theory at

imaginary chemical potential. In the first case, a phase transition of the fermionic partition function occurs when the saddle point reaches the boundary of the manifold. This does not happen for bosonic partition function when the saddle point manifold is noncompact, and the residual  $U(1)$  symmetry is always broken spontaneously. In the second example, an axial symmetry restoration phase transition occurs when the minimum of the effective potential trivializes. The two partition functions have the same phase diagram but may behave differently under analytical continuation.

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